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Minimizing shoreline pollution in rivers with tributaries

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If an unpolluted tributary joins a river less than about thirty channel breadths downstream of an effluent outlet, then the additional dilution can reduce the peak pollution level experienced at the shoreline. This paper identifies the optimal sites for steady discharges to take full advantage of the extra dilution.

1. Introduction

Contaminant plumes in rivers are extremely long and narrow (Fischer *et al.* 1979, figure 5.6). It can take of the order of a hundred channel breadths downstream for the contaminant to become well-mixed across a river (Yotsukura & Sayre 1976). Correspondingly, there is a long stretch of river for which the pollution levels are critically dependent upon the precise siting of an effluent outlet (Yotsukura & Cobb 1972; Smith 1987).

When a polluted river is joined by a clean tributary there will be eventual dilution of the contaminant, taking place over about a hundred channel breadths downstream of the junction. For sudden discharges Daish (1985) has shown that there is also extra longitudinal spreading, which he quantifies as a junction variance.

Here we consider steady discharges. If the contaminant plume has not yet spread across the polluted river, then the merging with the clean tributary can delay the contaminant reaching the riverbanks. Thus, not only is the eventual pollution level reduced, but so also is the peak pollution level experienced at the shoreline. The purpose of the present paper is to identify the optimal sites upstream of the junction so that the peak shoreline pollution level is minimized.

2. Flow geometry and equations

As explained by Daish (1985), and calculated in detail by Jovanovic & Officer (1985), the adjustment region for the flow at the river junction is short compared with the downstream lengthscale for mixing across the river. Thus, to deal with the junction region, we temporarily ignore transverse diffusion and we take the concentration to be carried with the flow (see figure 1). For simplicity, each reach of the river system is also assumed to be straight and we use conventional (x, y) coordinates along and across the flow. The matching becomes

$$c(x_0^-, y^*) = c(x_0^+, y), \tag{2.1}$$

where c is the concentration, x_0^- is just upstream of the junction, x_0^+ is just downstream of the junction, and the flow-following connects y^* to y across the

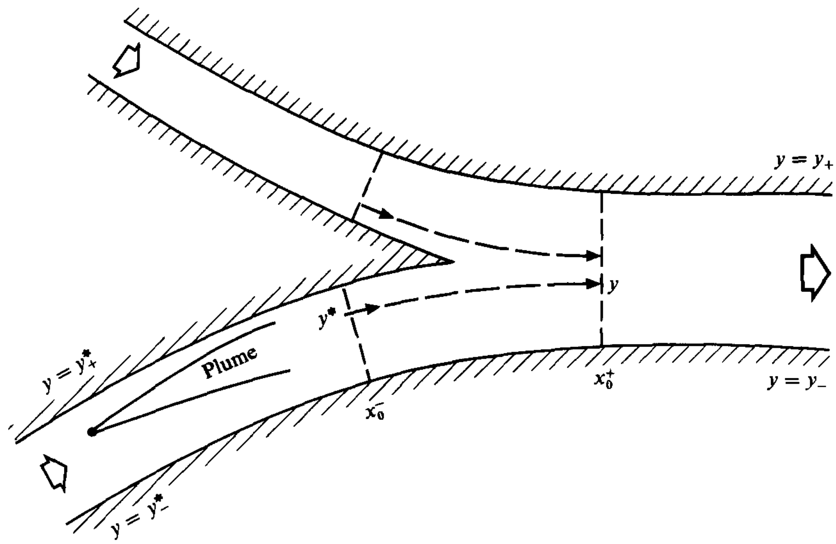

FIGURE 1. Sketch of a river junction with its flow regions. The contaminant plume would be much narrower and further upstream than indicated.

junction region. We use the convention employed by Daish (1985) of using asterisks to identify upstream quantities when there is any possibility of confusion.

For rivers with depth $h(y)$ much less than the breadth, the appropriate depth-average advection–diffusion equation is

$$hu\,\partial_x c = \partial_y(h\kappa\,\partial_y c), \tag{2.2a}$$

where $u(y)$ is the longitudinal velocity and $\kappa(y)$ the transverse diffusivity. At the Shorelines y_-^*, y_+^*, y_-, y_+ (see figure 1), the no-flux boundary condition is

$$h\kappa\,\partial_y c = 0. \tag{2.2b}$$

3. Avoiding the overshoot

In the downstream section of the river the concentration can be represented in terms of the eigenmodes $\phi_n(y)$:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(h\kappa\frac{\mathrm{d}}{\mathrm{d}y}\phi_n\right)+\mu_n hu\phi_n = 0, \tag{3.1a}$$

with

$$\frac{h\kappa\mathrm{d}\phi_n}{\mathrm{d}y} = 0 \quad \text{on } y = y_-,\, y_+, \tag{3.1b}$$

$$\int_{y_-}^{y_+} hu\phi_n^2\,\mathrm{d}y = \int_{y_-}^{y_+} hu\,\mathrm{d}y = Q. \tag{3.1c}$$

The zero mode is

$$\phi_0 = 1, \quad \mu_0 = 0. \tag{3.2a, b}$$

The solution for $c(x,y)$ can be written

$$c = c_0 + \sum_{n=1}^{\infty} c_n \exp\left[-\mu_n(x-x_0^+)\right]\phi_n(y), \tag{3.3a}$$

with
$$c_n = \frac{1}{Q}\int_{y_-}^{y_+} hu\phi_n\, c(x_0^+, y)\,\mathrm{d}y. \tag{3.3b}$$

The quantity Q is the volume flux rate of the flow in the downstream reach of the river.

Smith (1982, §4) explains that the only way to avoid an overshoot of concentration at either of the river banks is to require that

$$c_1 = 0. \tag{3.4}$$

(The first non-constant eigenfunction $\phi_1(y)$ has opposite signs at the two banks y_-, y_+. So, unless $c_1 = 0$, the concentration (3.3a) at large distances downstream will exceed the asymptote c_0 at one side of the channel). The elimination of ϕ_1 from the solution (3.3a) means that mixing across the flow is achieved as rapidly as possible, which is the design criterion suggested by Yotsukura & Cobb (1972).

4. Upstream extrapolation

Using flow-following we link the downstream eigenmode ϕ_1 to the upstream function Φ_1:

$$\Phi_1(x_0^-, y^*) = \phi_1(x_0^+, y). \tag{4.1}$$

Thus, in terms of the upstream depth and velocity profiles, the requirement (3.4) becomes

$$\int_{y_-^*}^{y_+^*} hu\Phi_1(x_0^-, y^*)\, c(x_0^-, y^*)\,\mathrm{d}y^* = 0. \tag{4.2}$$

The reciprocal theorem for advection–diffusion equations (F. B. Smith 1957) leads us to require that Φ_1 satisfies the reversed-flow problem:

$$-hu\,\partial_x \Phi_1 = \partial_y(h\kappa\,\partial_y \Phi_1), \tag{4.3a}$$

with
$$h\kappa\,\partial_y \Phi_1 = 0 \quad \text{on } y = y_-^*, y_+^*. \tag{4.3b}$$

With this upstream extrapolation of Φ_1, the condition (4.2) applies at all upstream values of x:

$$\int_{y_-^*}^{y_+^*} hu\,\Phi_1 c\,\mathrm{d}y^* = 0 \quad \text{for all } x \leqslant x_0^-. \tag{4.4}$$

It must be emphasized that Φ_1 is *not* an eigenmode of the upstream branch, but is an extrapolation of the downstream eigenmode ϕ_1.

For a point source, c is initially concentrated at a single point across the flow. Thus, the only way of achieving the result (4.4) is to position the source at a zero of $\Phi_1(x, y^*)$. This then defines the optimal discharge positions with respect to the far downstream pollution

$$\Phi_1(x, y^*) = 0. \tag{4.5}$$

Such a zero only exists if the single zero of Φ_1 transfers upstream into the appropriate reach of the river. This implies that it is generally better to make discharges in the tributary with the greater volume flow rate. By analogy with (3.3a, b) we can show that Φ_1 asymptotes upstream to the constant non-zero value

$$\frac{1}{Q^*}\int_{y_-^*}^{y_+^*} hu\Phi_1\,\mathrm{d}y^*. \tag{4.6}$$

Thus, the zero of Φ_1 ceases to exist far upstream of the junction. The asymptote (4.6) has the opposite sign to the part of ϕ_1 associated with the clean tributary. Hence, by the continuity of Φ_1, we can infer that the zero contour comes to the bank that connects to the tip of the junction.

5. Avoiding local pollution

Positioning a discharge at a riverbank may be compatible with minimizing pollution far downstream in another reach of the river, but is unacceptable in terms of the local pollution. Thus, we should ensure that the peak pollution level experienced in the first reach of the river is no greater than that in the downstream reach.

When $c_1 = 0$, the peak shoreline concentration far downstream is the asymptote

$$c_0 = \frac{1}{Q} \int_{y_-}^{y_+} huc \, \mathrm{d}y. \tag{5.1}$$

Since all the pollutant originates from one of the upstream reaches, this can be written in terms of the upstream concentration:

$$c_0 = \frac{Q^*}{Q} \frac{1}{Q^*} \int_{y_-^*}^{y_+^*} huc \, \mathrm{d}y^*. \tag{5.2}$$

The volume-flux ratio (Q^*/Q) is the improvement in pollution levels by the dilution. We require that in the upstream reach the shoreline concentration nowhere exceeds c_0.

Our main concern is with the bank that connects to the tip of the junction. We seek to ensure that the discharge is far enough away from that shoreline so that the shoreline concentration is less than c_0. The most stringent constraint stems from the furthest downstream location, i.e. the tip of the junction.

We again use the reciprocal theorem for advection–diffusion equations (F. B. Smith 1957; Smith 1983). For a shoreline discharge at the tip of the junction with the flow reversed, we define a Green function

$$-hu \, \partial_x G = \partial_y (h\kappa \, \partial_y G) + \delta(y - y_-^*) \, \delta(x - x_0^-) \, Q^*, \tag{5.3a}$$

with

$$h\kappa \, \partial_y G = 0 \quad \text{on } y = y_-^*, y_+^*. \tag{5.3b}$$

Upstream of the junction and downstream of the effluent outfall we have

$$\int_{y_-^*}^{y_+^*} huc \, G \, \mathrm{d}y^* = \text{constant}. \tag{5.4}$$

From the limiting cases in which c or G is concentrated at a single point we infer that the concentration c at the tip of the junction is given by

$$c(x_0^-, y_+^*) = G(x, y) \frac{1}{Q^*} \int_{y_-^*}^{y_+^*} huc \, \mathrm{d}y^*. \tag{5.5}$$

Thus, the permissible discharge locations are further from the shoreline than the (Q^*/Q) contour of the Green function $G\ (x, y)$.

The resulting optimal selection of discharge sites is where

$$\Phi_1(x,y) = 0, \quad G(x,y) \leqslant \frac{Q^*}{Q} \tag{5.6a, b}$$

The first condition ensures that far downstream the maximum shoreline concentration is c_0, while the second condition ensures that the same upper bound is satisfied in the upper reach. Both conditions enforce that the discharge sites to be positioned within a diffusion lengthscale of the junction. Otherwise the contaminant plume will have already reached the river banks in the upstream stretch of river, and the peak concentration level will occur before the junction is reached.

6. Illustrative example

To make the calculations as simple as possible, we model the depth, velocity and diffusivity as being proportional to each other:

$$h = H\left[\cos\left(\frac{\pi y}{2B}\right)\right]^{\frac{1}{2}}, \quad u = U\left[\cos\left(\frac{\pi y}{2B}\right)\right]^{\frac{1}{2}}, \quad \kappa = K\left[\cos\left(\frac{\pi y}{2B}\right)\right]^{\frac{1}{2}}, \tag{6.1a–c}$$

with

$$y_- = -B, \quad y_+ = B, \quad Q = 4\frac{HUB}{\pi}. \tag{6.1d–f}$$

The flow-following matching across the junction region gives

$$Q^*\left[1+\sin\left(\frac{\pi y^*}{2B^*}\right)\right] = Q\left[1+\sin\left(\frac{\pi y}{2B}\right)\right]. \tag{6.2}$$

In the downstream reach the first non-constant eigenfunction is

$$\phi_1 = (\tfrac{3}{2})^{\frac{1}{2}} \sin\left(\frac{\pi y}{2B}\right). \tag{6.3}$$

Hence, at the upstream end of the junction the upstream extrapolation Φ_1 has the starting value

$$\Phi_1 = -\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(1-\frac{Q^*}{Q}\right)+\left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{Q^*}{Q}\sin\left(\frac{\pi y}{2B^*}\right). \tag{6.4}$$

Conveniently, a series expansion for Φ_1 involves just two terms:

$$\Phi_1 = -\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(1-\frac{Q^*}{Q}\right)+\left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{Q^*}{Q}\sin\left(\frac{\pi y^*}{2B^*}\right)\exp\left(\frac{\pi^2 K^* x^*}{2U^* B^{*2}}\right) \quad \text{for } x < 0. \tag{6.5}$$

So, it is easy to find the zero contour (see figure 2):

$$\sin\left(\frac{\pi y^*}{2B^*}\right) = \left(\frac{Q}{Q^*}-1\right)\exp\left(\frac{-\pi^2 K^* x^*}{2U^* B^{*2}}\right). \tag{6.6}$$

The tip of the junction is at $(0, B^*)$. Upstream of the tip, the Green function $G(x,y)$ has the Legendre series representation

$$G = 1+\sum_{n=1}^{\infty}(2n+1)P_n\left(\sin\frac{\pi y^*}{2B^*}\right)\exp\left(\frac{n(n+1)\pi^2 K^* x}{4U^* B^{*2}}\right). \tag{6.7}$$

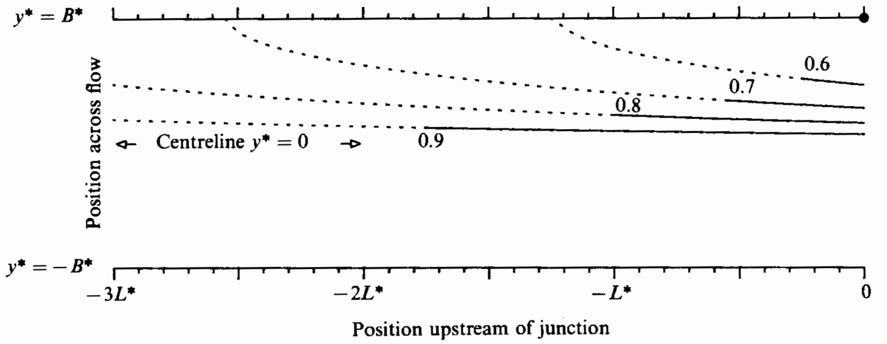

FIGURE 2. Optimal discharge positions upstream of a junction. The numbers indicate the ratio Q^*/Q of the total volume discharge rates in the upstream and downstream reaches of the river system. The diffusion scale L^* typically corresponds to about thirty times the channel breadth.

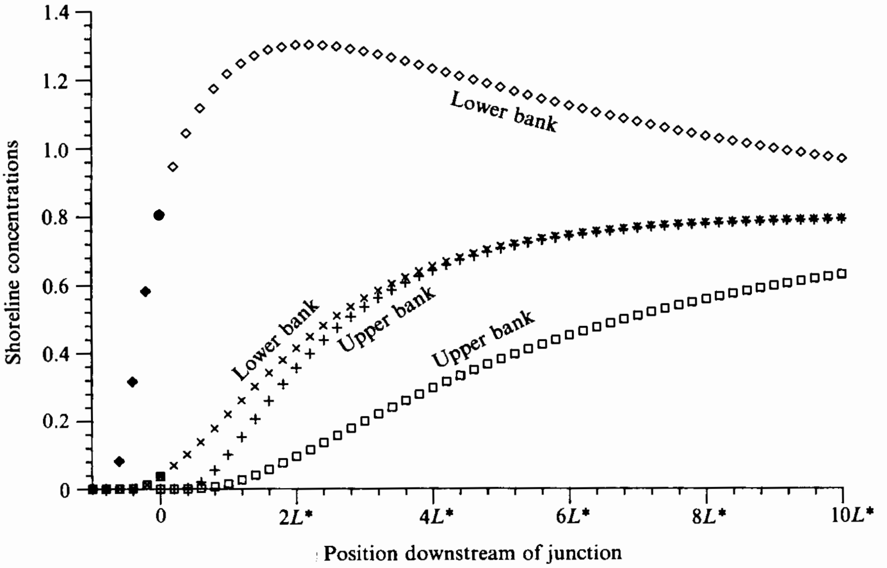

FIGURE 3. The concentration at the shorelines for an optimal discharge ($\times\times\times$, $+++$), and for a non-optimal discharge ($\diamond$, $\square$) the same distance from the opposite shore. There is a jump in position of the upper river bank which gives rise to a jump in the corresponding shoreline concentrations.

This formula permits us to ascertain which parts of the $\Phi_1 = 0$ contour satisfy the constraint

$$G < \frac{Q^*}{Q} \tag{6.8}$$

(indicated by the continuous curves in figure 2).

The diffusion lengthscale for transverse mixing can be estimated (with $n = 2$):

$$L^* = \frac{2U^* B^{*2}}{3\pi^2 K^*}. \tag{6.9}$$

For turbulent open-channel flows the magnitude K^* of the transverse diffusivity can be estimated:

$$K^* \approx 0.01\, H^* U^*. \tag{6.10}$$

Hence, the diffusion lengthscale becomes

$$L^* \approx \frac{200}{3\pi^2}\left(\frac{B^*}{H^*}\right)B^* \approx 6.75\left(\frac{B^*}{H^*}\right)B^*. \tag{6.11}$$

If the aspect ratio $B{:}H$ is of order 10:1 then L^* corresponds to over thirty channel breadths ($60B^*$) upstream of the junction.

Figure 3 illustrates the difference in shoreline concentrations for optimal and non-optimal discharges. The discharges take place a distance L^* upstream of a junction with flux ratio $Q^*/Q = 0.8$. The non-optimal discharge is at the same distance 0.77 B^* from the upper bank as the optimal discharge is from the lower bank. Hence upstream of the junction the shoreline concentrations are merely swapped around. It is downstream of the junction that the shortcomings of the non-optimal (lower) discharge becomes apparent. For a distance of several hundred channel breadths downstream of the junction, there is a substantial overshoot in the concentrations experienced along the nearest bank. (To fix the downstream lengthscale we have assumed that the aspect ratio $B{:}H$ is the same as in the upstream reaches.)

7. Meandering streams

It is in the physics that there is the most profound difference between straight and meandering streams. The curvature can induce secondary flows which greatly augment the transverse mixing (Rozovskii 1961). However, if the water depth is much less than the channel breadth, then this augmented mixing can be allowed for merely by including a secondary-flow contribution in the transverse mixing coefficient κ (Fischer 1969).

Mathematically, the effects of meandering can be accommodated by the use of flow-following coordinates (Smith 1982). The x-dependence of the flow quantities h, κ, u means that the eigenmodes are not self-adjoint. The concentration $c(x, y)$ is expanded in terms of downstream modes $\phi_n(x, y)$, with coefficients c_n which depend upon the adjoint upstream mode $\hat{\phi}_n(x, y)$. Hence, at a junction it is the upstream mode $\hat{\phi}_1(x, y)$ that has to be extrapolated upstream. The most significant difference from the straight-river case is that the larger κ-values will make the diffusion lengthscale L^* shorter. So, the discharges will have to be closer to the junction to take full advantage of the extra dilution.

8. More than one junction

Another complication of real river systems is that on the diffusion lengthscale there can be more than one tributary joining a river. In principle, the necessary generalization of the above analysis is straightforward. The eigenmode ϕ_1 for the furthest downstream reach needs to be extrapolated upstream. The zero contour

$$\Phi_1(x, y) = 0 \tag{8.1}$$

defines the optimal discharge positions with respect to the far downstream pollution. The upstream penetration of this contour will be of the order of the diffusion lengthscale. Also, by continuity, the contour will only enter at most one of the tributaries.

To avoid local pollution we again ensure that the discharge is far enough away

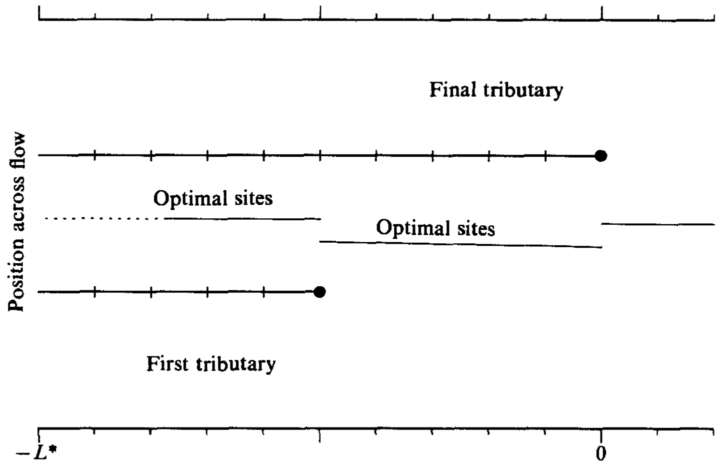

FIGURE 4. Optimal discharge positions for the merging of three identical tributaries.

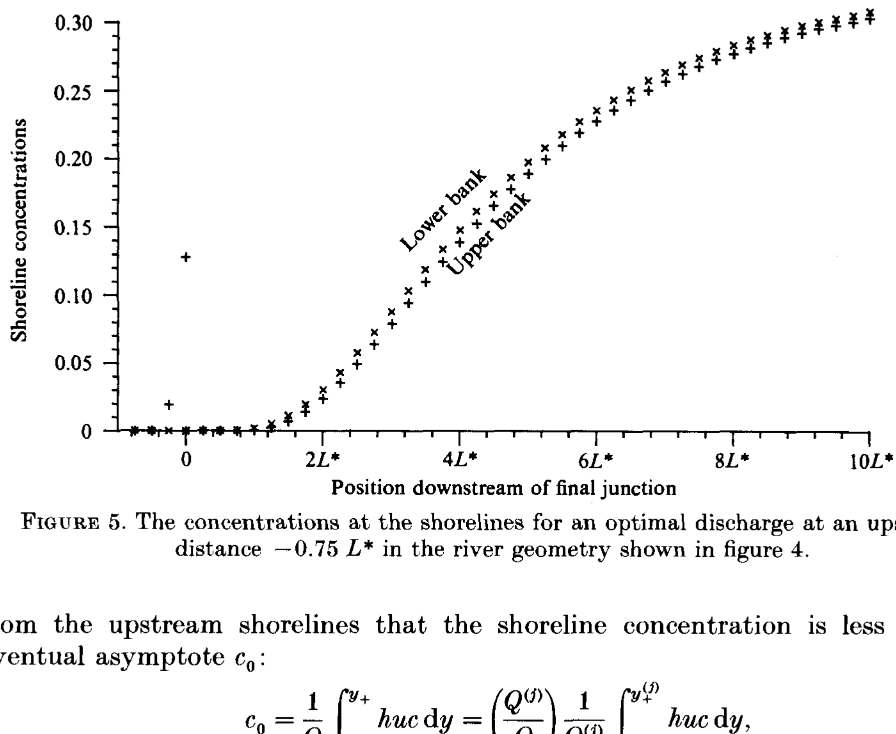

FIGURE 5. The concentrations at the shorelines for an optimal discharge at an upstream distance $-0.75\ L^*$ in the river geometry shown in figure 4.

from the upstream shorelines that the shoreline concentration is less than the eventual asymptote c_0:

$$c_0 = \frac{1}{Q}\int_{y_-}^{y_+} huc\,\mathrm{d}y = \left(\frac{Q^{(j)}}{Q}\right)\frac{1}{Q^{(j)}}\int_{y_-^{(j)}}^{y_+^{(j)}} huc\,\mathrm{d}y, \tag{8.2}$$

where $Q^{(j)}$ is the volume flow rate for the corresponding reach of the river system. For the respective tips of the junctions we define Green functions $G^{(k)}$ with tip source strength $Q^{(k)}$. The condition for there to be no excessive pollution upstream of the jth tip is

$$G^{(k)}(x,y) < \frac{Q^{(j)}}{Q} \tag{8.3}$$

along the contour (8.1).

As an illustrative example we consider a river system with the same depth, velocity, and transverse diffusivity profiles as used in §6. The width and depth of each stretch of the river system are assumed to be proportional to the local volume flow rate. Figure 4 shows the optimal discharge sites for the merging of three identical tributaries. After each of the junctions the redistribution of the bulk flow gives rise to the abrupt shift in the optimal position. The mixing length L^* is defined as in (6.9) in terms of the furthest upstream flow conditions.

When there are such multiple junctions, the benefits to be gained from optimal discharge siting can be much greater than for single junctions. Figure 5 shows the shoreline concentration for an optimal discharge at

$$x = -0.75\,L^*, \quad y = 0.067\,B^*. \tag{9.1}$$

The peak shoreline concentration is one third of the best that could have been achieved in any one of the tributaries alone.

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